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Newman–Penrose conserved quantities in Born–Infeld electrodynamics†

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Abstract. The Newman–Penrose conserved quantities for the Born–Infeld nonlinear electromagnetic theory and the Einstein–Born–Infeld theory are shown to be precisely the same as those for the Maxwell and Einstein–Maxwell theories respectively. It is suggested that this occurrence is due to the particular nature of the nonlinearity of the Born–Infeld field equations.

1. Introduction

Newman and Penrose (1965, 1968) and Exton *et al* (1969) have discovered new conserved quantities in both the Maxwell and Einstein–Maxwell theories. In this paper the new conserved quantities for the Born–Infeld nonlinear electromagnetic theory (Born 1934, Born and Infeld 1934) and the combined Einstein–Born–Infeld theory are shown to be precisely the same as those for the Maxwell and Einstein–Maxwell theories respectively. These, at first sight, slightly surprising results become more understandable when it is realized that the equations that give rise to the conserved quantities are derived by equating coefficients of certain powers of the inverse of a radial parameter in the expansion of the field equations. In the Born–Infeld and Einstein–Born–Infeld theories the relevant equations turn out not to involve any non-Maxwell or non-Einstein–Maxwell terms. It is suggested that this occurrence is due to the particular nature of the nonlinearity in the Born–Infeld field equations.

A knowledge of the Newman–Penrose (to be referred to as NP) spin-coefficient formalism (Newman and Penrose 1962), the properties of the operator $\eth$ and the spin-weighted spherical harmonics (Newman and Penrose 1966, Goldberg *et al* 1967) will be assumed. The Born–Infeld field equations were derived in the NP formalism in a previous paper (Chellone 1971 to be referred to as I).

2. Born–Infeld theory

The field equations for Born–Infeld electrodynamics in flat space–time are (see, for example, I)

$$F^{\mu\nu}{}_{,\nu}^{+} = \tfrac{1}{4}b^{-2}F^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma,\nu} - \tfrac{1}{2}FF^{\mu\nu}{}_{,\nu} \tag{2.1}$$

† A referee has pointed out that some of the results of this paper have been independently discovered by J R Porter, Mathematics Department, University of Pittsburgh, Pa, USA in a paper to appear in *the Proceedings of the Cambridge Philosophical Society.*

where $F^{\mu\nu}$ is the electromagnetic field tensor, b is a constant and

$$F^{\mu\nu+} = \tfrac{1}{2}(F^{\mu\nu}+\mathrm{i}F^{*\mu\nu}).$$

These equations can be rewritten as four equations in ϕ (tetrad components of the $F^{\mu\nu}$) and spin coefficients (see I for details). If ϕ_0, ϕ_1 and ϕ_2 are assumed to have expansions in positive powers of r^{-1}, the results of paper I and the 'peeling' properties of ϕ_0, ϕ_1 and ϕ_2 lead to the expansions having the form

$$\begin{aligned}\phi_0 &= \phi_0^0 r^{-3}+\phi_0^1 r^{-4}+\mathrm{O}(r^{-5})\\ \phi_1 &= \phi_1^0 r^{-2}+\mathrm{O}(r^{-3})\\ \phi_2 &= \phi_2^0 r^{-1}+\mathrm{O}(r^{-2}).\end{aligned} \tag{2.2}$$

The field equations can now be written symbolically as

$$-\frac{\partial\phi_1}{\partial u}+\frac{1}{2}\frac{\partial\phi_1}{\partial r}+r^{-1}\phi_1-2^{-1/2}r^{-1}\eth\phi_2 = \mathrm{O}(r^{-6},b^{-2}) \tag{2.3}$$

$$\frac{\partial\phi_1}{\partial r} + 2^{-1/2}r^{-1}\bar{\eth}\phi_0+2r^{-1}\phi_1 = \mathrm{O}(r^{-7},b^{-2}) \tag{2.4}$$

$$\frac{\partial\phi_2}{\partial r}+2^{-1/2}r^{-1}\bar{\eth}\phi_1+r^{-1}\phi_2 = \mathrm{O}(r^{-6},b^{-2}) \tag{2.5}$$

$$-\frac{\partial\phi_0}{\partial u}+\frac{1}{2}\frac{\partial\phi_0}{\partial r}+\frac{1}{2}r^{-1}\phi_0-2^{-1/2}r^{-1}\eth\phi_1 = \mathrm{O}(r^{-6},b^{-2}) \tag{2.6}$$

where, owing to the complexity of the expressions on the right hand sides only the order of their r and b dependences has been shown.

3. The new conserved quantities

Using a similar approach to the standard way of dealing with the spin-coefficient Maxwell equations (the limit of equations (2.3)–(2.6) as $b \to \infty$ (Janis and Newman 1965)) multiplying (2.4) by r^2 and integrating yields

$$\phi_1 = r^{-2}\int 2^{-1/2}r\bar{\eth}\phi_0\,\mathrm{d}r+\mathrm{O}(r^{-6})$$

which, on expanding ϕ_0 becomes

$$\phi_1 = r^{-2}\phi_1^0+2^{-1/2}\bar{\eth}\sum_{n\geqslant 1}\left(\frac{\phi_0^{n-1}}{nr^{n+2}}\right)+\mathrm{O}(r^{-6}).$$

Multiplying (2.5) by r and integrating leads to

$$\phi_2 = r^{-1}\phi_2^0+2^{-1/2}r^{-2}\bar{\eth}\phi_1^0+\frac{1}{2}\bar{\eth}^2\sum_{n\geqslant 1}\left(\frac{\phi_0^{n-1}}{n(n+1)r^{n+2}}\right)+\mathrm{O}(r^{-5}).$$

Substituting these back in the other two field equations and equating coefficients of r^{-2} in (2.3) gives

$$\dot{\phi}_1^0 = -2^{-1/2}\eth\phi_2^0 \tag{3.1}$$

while equating coefficients of r^{-4} in (2.6), yields, after using the commutation relation

$$(\bar{\eth}\eth-\eth\bar{\eth})\eta = 2s\eta$$

where η has spin-weight s

$$\dot{\phi}_0^1 = \tfrac{1}{2}\bar{\eth}\eth\phi_0^0. \tag{3.2}$$

To obtain the law of charge conservation, equation (3.1) is multiplied by ${}_0\bar{Y}_{00}$ and integrated over the sphere

$$\frac{\mathrm{d}}{\mathrm{d}u}\int {}_0\bar{Y}_{00}\phi_1^0\,\mathrm{d}S = -2^{-1/2}\int {}_0\bar{Y}_{00}\eth\phi_2^0\,\mathrm{d}S.$$

The right hand side vanishes as a result of the properties of $\eth$ (Newman and Penrose 1968) and so

$$\int {}_0\bar{Y}_{00}\phi_1^0\,\mathrm{d}S = \text{constant.}$$

This constant is proportional to a_0 of paper I, where a_0 was interpreted as the electric charge plus i times the 'magnetic' charge. As the effect of the nonlinear Born–Infeld terms is not felt until coefficients of r^{-5} are considered, this charge conservation law is precisely the same as the law in Maxwell theory. The equation that gives rise to the NP conserved quantities (3.2) is also of sufficiently low r^{-1} dependence to not involve any nonlinear terms, so the NP conservation laws are exactly the same as those that occur in Maxwell theory. They are obtained (Newman and Penrose 1968) by multiplying (3.2) by ${}_1\bar{Y}_{1m}$, integrating over the sphere and using a property of $\eth$, leaving as conserved quantities

$$F_m = \int {}_1\bar{Y}_{1m}\phi_0^1\,\mathrm{d}S \qquad m = -1, 0, 1.$$

The real and imaginary parts of F_{-1}, F_0 and F_1 are the six new conserved quantities for both the Maxwell and Born–Infeld fields which remain conserved when gravitational fields are included.

The usual interpretation of the F_m in terms of arbitrary incoming dipole radiation (Newman and Penrose 1968) carries through in the Born–Infeld case with the following modifications.

In general, arbitrary Maxwell dipole radiation in Born–Infeld theory will give rise to 'tail' terms of order b^{-2} and r^{-3} (paper I), and these outgoing tail terms will contribute to the F_m as well as the incoming dipole field. Hence the interpretation of the F_m is that they represent essentially the coefficient of r^{-4} in the expansion of ϕ_0 where ϕ_0 is an approximate, mainly incoming, Born–Infeld dipole solution (details of solutions of this type were given in paper I).

It is perhaps remarkable that one reason why the new conserved quantities exist in the Born–Infeld theory is due to the high degree of nonlinearity present in the right hand sides of the field equations. If they were quadratic in the ϕ instead of cubic then the right hand sides of equation (2.6) would usually involve terms of order r^{-4} and so the equation that gives rise to the new conserved quantities, (3.2) would not, in general, occur in a suitable form.

4. Newman–Penrose conserved quantities for the Einstein–Born–Infeld field

The NP conserved quantities for the Einstein–Born–Infeld field are precisely the same as those for the Einstein–Maxwell field. However in order to see this it is necessary to use the full NP formalism instead of the comparatively simple flat space–time form. The only quantities required which have not been given previously are the tetrad components of the Ricci tensor, Φ_{mn}, $m, n = 0, 1, 2$.

The energy-momentum tensor for the Einstein–Born–Infeld field is

$$T_{\mu\nu} = k[g_{\mu\nu}b^2\{1-(1+F)^{1/2}\}+F_{\mu\lambda}F_\nu{}^\lambda(1+F)^{-1/2}]$$

whence

$$R_{\mu\nu} = k'(1+F)^{-1/2}[F_{\mu\lambda}F_\nu{}^\lambda - g_{\mu\nu}b^2\{(1+F)^{1/2}-1\}]$$

where k and k' are constants. When the ϕ have been expanded in inverse powers of r according to (2.2) the tetrad components of the Ricci tensor can be found (table 1).

Table 1. Tetrad components of the Ricci tensor

Tetrad component	Order of largest Born–Infeld term
$\Phi_{00} \equiv -\frac{1}{2}R_{00} = kk'\{\phi_0^0\bar\phi_0^0 r^{-6}+O(r^{-7})\}$	$b^{-2}r^{-10}$
$\Phi_{11} \equiv -\frac{1}{4}(R_{01}+R_{23}) = kk'\{\phi_1^0\bar\phi_1^0 r^{-4}+O(r^{-5})\}$	$b^{-2}r^{-8}$
$\Phi_{01} \equiv -\frac{1}{2}R_{02} = kk'\{\phi_0^0\bar\phi_1^0 r^{-5}+O(r^{-6})\}$	$b^{-2}r^{-9}$
$\Phi_{12} \equiv -\frac{1}{2}R_{12} = kk'\{\phi_1^0\bar\phi_2^0 r^{-3}+O(r^{-4})\}$	$b^{-2}r^{-7}$
$\Phi_{10} \equiv -\frac{1}{2}R_{03} = kk'\{\phi_1^0\bar\phi_0^0 r^{-5}+O(r^{-6})\}$	$b^{-2}r^{-9}$
[illegible] $\equiv -\frac{1}{2}R_{13} = kk'\{\phi_2^0\bar\phi_1^0 r^{-3}+O(r^{-4})\}$	$b^{-2}r^{-7}$
[illegible] $\equiv -\frac{1}{2}R_{22} = kk'\{\phi_0^0\bar\phi_2^0 r^{-4}+O(r^{-5})\}$	$b^{-2}r^{-8}$
[illegible] $\equiv -\frac{1}{2}R_{11} = kk'\{\phi_2^0\bar\phi_2^0 r^{-2}+O(r^{-3})\}$	$b^{-2}r^{-6}$
$\Phi_{20} \equiv -\frac{1}{2}R_{33} = kk'\{\phi_2^0\bar\phi_0^0 r^{-4}+O(r^{-5})\}$	$b^{-2}r^{-8}$

The definition of the Φ_{mn} in terms of the Ricci tensor components is consistent with that given in Newman and Penrose (1962).

As has been anticipated, the terms involving powers of b^{-2} (the Born–Infeld terms) have higher r^{-1} dependence than would be required to contribute to the conserved quantities. Accordingly the conserved quantities in the Einstein–Maxwell theory are precisely those that are conserved in the Einstein–Born–Infeld theory, that is they are the six quantities described in the previous section (which are still conserved in the presence of a gravitational field) plus the ten quantities for the Einstein–Maxwell field. The calculations that show that the quantities referred to are in fact conserved will not, for reasons of space, be reproduced here. They can be found in Exton *et al* (1969) and Exton (1967).

The ten conserved quantities are:

$$G_m = \int {}_2\bar Y_{2m}\{2\psi_0^1+2(2^{3/2}\phi_0^1+F)\bar\eth^{-1}(2\bar\phi_1^0-\bar E)-2(2\bar\phi_1^0+\bar E)\bar\eth^{-1}(2^{3/2}\phi_0^1-F)\}\,dS$$

where ψ_0^1 is the coefficient of r^{-4} in the expansion of ψ_0 and $\bar{\eth}^{-1}$ is an operator defined by

$$\bar{\eth}^{-1}(2\bar{\phi}_1^0 - \bar{E}) = -\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\{l(l+1)\}^{-1/2}\alpha_{lm}\,{}_1Y_{lm}$$

$$\bar{\eth}^{-1}(2^{3/2}\phi_0^1 - F) = -\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\{(l-1)(l+2)\}^{-1/2}\beta_{lm}\,{}_2Y_{lm}$$

where

$$\alpha_{lm} \equiv 2\int {}_0\bar{Y}_{lm}\bar{\phi}_1^0\,\mathrm{d}S$$

$$\beta_{lm} \equiv 2^{3/2}\int {}_1\bar{Y}_{lm}\phi_0^1\,\mathrm{d}S$$

$$E \equiv \bar{e}_0 Y_{00} \qquad e = 2\int {}_0\bar{Y}_{00}\phi_1^0\,\mathrm{d}S = \text{constant}$$

$$F = \sum_{m=-l}^{l} F_m\,{}_1Y_{1m}.$$

5. Conclusion

The NP conserved quantities have been shown to be the same for Born–Infeld theory as Maxwell theory and the same for Einstein–Born–Infeld theory as Einstein–Maxwell theory. The fact that this happens is attributed to the high degree of nonlinearity of the Born–Infeld field equations. It is conjectured that in any electromagnetic theory of the type

$$M^{\mu}(\phi) = B^{\mu}(\phi) \qquad \mu = 0, 1, 2, 3$$

where $M^{\mu}(\phi) = 0$ would give the source-free Maxwell equations, the NP quantities would be conserved if all the $B^{\mu}(\phi)$ contained cubic or higher powers of the ϕ, but not conserved if all the $B^{\mu}(\phi)$ were linear or quadratic.

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